

Contents

PART ONE

General Topology 1

CHAPTER I

Sets 3

§1. Some Basic Terminology 3

§2. Denumerable Sets 7

§3. Zorn's Lemma 10

CHAPTER II

Topological Spaces 17

§1. Open and Closed Sets 17

§2. Connected Sets 27

§3. Compact Spaces 31

§4. Separation by Continuous Functions 40

§5. Exercises 43

CHAPTER III

Continuous Functions on Compact Sets 51

§1. The Stone–Weierstrass Theorem 51

§2. Ideals of Continuous Functions 55

§3. Ascoli's Theorem 57

§4. Exercises 59

PART TWO

Banach and Hilbert Spaces 63

CHAPTER IV

Banach Spaces 65

§1. Definitions, the Dual Space, and the Hahn–Banach Theorem 65

§2. Banach Algebras 72

§3. The Linear Extension Theorem 75

§4. Completion of a Normed Vector Space 76

§5. Spaces with Operators 81

Appendix: Convex Sets 83

1. The Krein–Milman Theorem 83

2. Mazur's Theorem 88

§6. Exercises 91

CHAPTER V

Hilbert Space 95

§1. Hermitian Forms 95

§2. Functionals and Operators 104

§3. Exercises 107

PART THREE

Integration 109

CHAPTER VI

The General Integral 111

§1. Measured Spaces, Measurable Maps, and Positive Measures 112

§2. The Integral of Step Maps 126

§3. The L^1 -Completion 128

§4. Properties of the Integral: First Part 134

§5. Properties of the Integral: Second Part 137

§6. Approximations 147

§7. Extension of Positive Measures from Algebras to σ -Algebras 153

§8. Product Measures and Integration on a Product Space 158

§9. The Lebesgue Integral in \mathbf{R}^p 166

§10. Exercises 172

CHAPTER VII

Duality and Representation Theorems 181§1. The Hilbert Space $L^2(\mu)$ 181§2. Duality Between $L^1(\mu)$ and $L^\infty(\mu)$ 185

§3. Complex and Vectorial Measures 195

§4. Complex or Vectorial Measures and Duality 204

§5. The L^p Spaces, $1 < p < \infty$ 209

§6. The Law of Large Numbers 213

§7. Exercises 217

CHAPTER VIII

Some Applications of Integration	223
§1. Convolution	223
§2. Continuity and Differentiation Under the Integral Sign	225
§3. Dirac Sequences	227
§4. The Schwartz Space and Fourier Transform	236
§5. The Fourier Inversion Formula	241
§6. The Poisson Summation Formula	243
§7. An Example of Fourier Transform Not in the Schwartz Space	244
§8. Exercises	247

CHAPTER IX

Integration and Measures on Locally Compact Spaces	251
§1. Positive and Bounded Functionals on $C_c(X)$	252
§2. Positive Functionals as Integrals	255
§3. Regular Positive Measures	265
§4. Bounded Functionals as Integrals	267
§5. Localization of a Measure and of the Integral	269
§6. Product Measures on Locally Compact Spaces	272
§7. Exercises	274

CHAPTER X

Riemann–Stieltjes Integral and Measure	278
§1. Functions of Bounded Variation and the Stieltjes Integral	278
§2. Applications to Fourier Analysis	287
§3. Exercises	294

CHAPTER XI

Distributions	295
§1. Definition and Examples	295
§2. Support and Localization	299
§3. Derivation of Distributions	303
§4. Distributions with Discrete Support	304

CHAPTER XII

Integration on Locally Compact Groups	308
§1. Topological Groups	308
§2. The Haar Integral, Uniqueness	313
§3. Existence of the Haar Integral	319
§4. Measures on Factor Groups and Homogeneous Spaces	322
§5. Exercises	326

PART FOUR

Calculus	329
-----------------------	-----

CHAPTER XIII

Differential Calculus	331
§1. Integration in One Variable	331
§2. The Derivative as a Linear Map	333
§3. Properties of the Derivative	335
§4. Mean Value Theorem	340
§5. The Second Derivative	343
§6. Higher Derivatives and Taylor's Formula	346
§7. Partial Derivatives	351
§8. Differentiating Under the Integral Sign	355
§9. Differentiation of Sequences	356
§10. Exercises	357

CHAPTER XIV

Inverse Mappings and Differential Equations	360
§1. The Inverse Mapping Theorem	360
§2. The Implicit Mapping Theorem	364
§3. Existence Theorem for Differential Equations	365
§4. Local Dependence on Initial Conditions	371
§5. Global Smoothness of the Flow	376
§6. Exercises	379

PART FIVE

Functional Analysis	385
----------------------------------	-----

CHAPTER XV

The Open Mapping Theorem, Factor Spaces, and Duality	387
§1. The Open Mapping Theorem	387
§2. Orthogonality	391
§3. Applications of the Open Mapping Theorem	395

CHAPTER XVI

The Spectrum	400
§1. The Gelfand–Mazur Theorem	400
§2. The Gelfand Transform	407
§3. C^* -Algebras	409
§4. Exercises	412

CHAPTER XVII

Compact and Fredholm Operators	415
§1. Compact Operators	415
§2. Fredholm Operators and the Index	417
§3. Spectral Theorem for Compact Operators	426
§4. Application to Integral Equations	432
§5. Exercises	433

CHAPTER XVIII

Spectral Theorem for Bounded Hermitian Operators 438

- §1. Hermitian and Unitary Operators 438
- §2. Positive Hermitian Operators 439
- §3. The Spectral Theorem for Compact Hermitian Operators 442
- §4. The Spectral Theorem for Hermitian Operators 444
- §5. Orthogonal Projections 449
- §6. Schur's Lemma 452
- §7. Polar Decomposition of Endomorphisms 453
- §8. The Morse–Palais Lemma 455
- §9. Exercises 458

CHAPTER XIX

Further Spectral Theorems 464

- §1. Projection Functions of Operators 464
- §2. Self-Adjoint Operators 469
- §3. Example: The Laplace Operator in the Plane 476

CHAPTER XX

Spectral Measures 480

- §1. Definition of the Spectral Measure 480
- §2. Uniqueness of the Spectral Measure:
the Titchmarsh–Kodaira Formula 485
- §3. Unbounded Functions of Operators 488
- §4. Spectral Families of Projections 490
- §5. The Spectral Integral as Stieltjes Integral 491
- §6. Exercises 492

PART SIX

Global Analysis 495

CHAPTER XXI

Local Integration of Differential Forms 497

- §1. Sets of Measure 0 497
- §2. Change of Variables Formula 498
- §3. Differential Forms 507
- §4. Inverse Image of a Form 512
- §5. Appendix 516

CHAPTER XXII

Manifolds 523

- §1. Atlases, Charts, Morphisms 523
- §2. Submanifolds 527
- §3. Tangent Spaces 533
- §4. Partitions of Unity 536
- §5. Manifolds with Boundary 539
- §6. Vector Fields and Global Differential Equations 543

CHAPTER XXIII

Integration and Measures on Manifolds 547

§1. Differential Forms on Manifolds 547

§2. Orientation 551

§3. The Measure Associated with a Differential Form 553

§4. Stokes' Theorem for a Rectangular Simplex 555

§5. Stokes' Theorem on a Manifold 558

§6. Stokes' Theorem with Singularities 561

Bibliography 569

Table of Notation 572

Index 575