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This book is an attempt to provide a systematic course of the formal theory of fuzzy logic. We made a lot of effort to be precise, but at the same time to explain the motivation and interpretation of all the results and, if possible, to accompany the theory by examples.

There are a lot of other books on various aspects of fuzzy logic. Our book is more specific from the point of view of several aspects. First, it is based on logical formalism demonstrating that fuzzy logic is a well developed logical theory. Second, it includes the theory of functional systems in fuzzy logic, which provide explanation of what, and how can be represented by formulas of fuzzy logic called. Third, except for the generalization of the classical way of interpretation within the environment of fuzzy sets constructed over algebraic sets, it also presents much more general interpretation of fuzzy logic within the environment of other proper categories of fuzzy sets extending over the space theory, or even generalizing the latter. Last but not least, the leading philosophical point of view is presentation of fuzzy logic especially as the theory of vagueness as well as the theory of the common-sense human reasoning, which is based on the use of natural language, the distinguished feature of which is vagueness of its semantics.

We wrote the book to be read by people interested in fuzzy logic and related areas, and also by logicians, mathematicians and computer scientists interested in mathematical aspects of fuzzy logic. It can be used in special courses of fuzzy logic, artificial and computational intelligence, control and operational university studies, in advanced courses on various applications involving fuzzy logic, decision making and others.

The book is divided into eight chapters. The first chapter is introductory and it provides motivation for the development of fuzzy logic, describes its structure and outlines its potential for applications. Fuzzy logic is then divided into two main areas: (a) which is a special many-valued logic aiming at description of the vagueness phenomenon and that, in broader sense, is the theory of vagueness; (b) which is a theory of natural human reasoning based on the use of natural language. It is argued that characteristics of the vagueness phenomenon is fundamental for further development of fuzzy logic as well as its applications.