Contents

Preface v

Chapter 1 Symmetries in ODE's and PDE's 1

1.1 Euclidean symmetries : the basic notions 3
1.1.1 The Euclidean group 4
1.1.2 The closed subgroups of $\mathbf{O}(2)$ and $\mathbf{O}(3)$ 6
1.1.3 Lattice groups and lattice symmetries 8
1.2 Differential systems of physics and their symmetries 10
1.2.1 Examples of ODEs with symmetry : coupled oscillators 11
1.2.2 Elasticity : buckling problems 14
1.2.3 Reaction-diffusion equations 16
1.2.4 Hydrodynamical models 20
1.2.5 The symmetry of classical differential operators 31
1.3 Exercises 33

Chapter 2 Equivariant bifurcations, a first look 35

2.1 Group actions on Banach spaces 35
2.2 The equivariant Lyapunov-Schmidt decomposition 41
2.3 The equivariant branching lemmas 49
2.3.1 The steady-state equivariant branching lemma 49
2.3.2 The equivariant branching lemma for symmetry groups acting in $\mathbb{R}^2$ and $\mathbb{R}^3$ 52
2.4 The equivariant Hopf bifurcation 55
2.4.1 Hopf bifurcation as a symmetry-breaking bifurcation problem 56

2.4.2 The equivariant Hopf branching lemma 60
2.5 Exercises 66

Chapter 3 Invariant manifolds and normal forms 71
3.1 Invariant manifolds for autonomous ODE's 73
3.2 The normal form reduction 79
3.3 Center manifolds and normal forms in bifurcation problems . . 81
3.3.1 Center manifold and normal form for a parameter-dependent ODE 82
3.3.2 Effective computation of the center manifold and normal form 84
3.4 Center manifolds for partial differential equations 88
3.4.1 Evolution equations in Banach spaces and center manifolds 88
3.4.2 An example: the Swift-Hohenberg equation on the sphere 93
3.5 Exercises 99

Chapter 4 Linear Lie Group Actions 103
4.1 Introduction 103
4.2 Lie groups 105
4.3 Induced actions 113
4.4 Representations 117
4.5 Characters 122
4.6 Representations of some continuous groups 134
4.6.1 The group $\mathbf{SO}(2)$ 134
4.6.2 The group $\mathbf{O}(2)$ 134
4.6.3 The group $\mathbf{O}(3)$ 134
4.6.4 The group $\mathbf{D}_m \ltimes \mathbb{T}^2$ 138
4.7 A remark on non-compact groups 142
4.8 Infinite dimensional representations 142
4.9 Generic one parameter families of equivariant linear maps . . . 145
4.10 Geometry of representations 148
4.11 The equivariant Whitney embedding theorem 152
4.12 Exercises 153

Chapter 5 The equivariant structure of bifurcation equations 155
5.1 Some introductory examples 156
5.1.1 $\mathbb{Z}_2$ - equivariance 156
5.1.2 $\mathbb{Z}_n$ - equivariance 157
5.1.3 $\mathbf{D}_n$ - equivariance 158
5.1.4 Structure of $\mathbf{O}(2)$-equivariant bifurcation equations . . . 159
5.2 General structure of bifurcation equations 160
5.2.1 The ring of invariant polynomials 160
5.2.2 The module of equivariant polynomial maps 163
5.3 The smooth case 165
5.4 Finding the degrees of generators: Molien series 168
5.5 Computation of invariants and equivariants 172
5.5.1 The case of finite groups 172
5.5.2 Continuous groups 173
5.6 Another Example 180
5.7 Normal hyperbolicity in equivariant problems 182
5.8 Exercises 185

Chapter 6 Reduction techniques for equivariant systems 187
6.1 The invariant sphere theorem 188
6.2 The orbit space reduction 193
6.2.1 The structure of the orbit space 196
6.2.2 Local structure of the orbit space 201
6.2.3 Projection of equivariant vector fields 203
6.2.4 Bifurcation and dynamics in the orbit space: an example 206
6.3 Exercises 211

Chapter 7 Relative equilibria and relative periodic orbits 213
7.1 Traveling waves, rotating waves and more 213
7.2 The flow on a relative equilibrium or periodic orbit 218
7.3 Decomposing equivariant flows near relative equilibria 225
7.4 Stability and bifurcation from relative equilibria 227
7.5 The equivariant Poincaré map for the flow near a RPO 231
7.6 Stability and bifurcation from RPO's 235
7.7 Bifurcation for equivariant maps 237
7.8 Non-compact groups, dynamics and applications 247
7.8.1 Non-spatially periodic solutions in extended systems . . 247

7.8.2 Actions of non-compact Lie groups 253
7.8.3 Slices and Tubes 255
7.8.4 An extension of Krupa's result and skew product flows . 256
7.8.5 Characterization of compact subgroups of $\mathbf{E}(n)$ 256
7.8.6 Relative equilibria, meanders and drifts 258
7.9 Exercises 260

Chapter 8 Bifurcations in Equivariant Systems 263
8.1 The Equivariant Branching Lemma revisited 264
8.1.1 The steady state case 265
8.1.2 Hopf bifurcation 268
8.1.3 How to detect Hopf bifurcations 270
8.2 Bifurcation with maximal isotropy 273
8.3 Bifurcation with non-maximal isotropy 280
8.3.1 Orbits, fixed point spaces and subconjugacy 282
8.3.2 Analytical methods 284
8.3.3 Topological methods 289
8.3.4 The topological degree in finite dimensional spaces . . . 292
8.4 The computation of relative equilibria and RPO's 294
8.5 Related topics 298
8.5.1 Equivariant topological degree 298
8.5.2 Gradient systems 299
8.5.3 Equivariant transversality theory 300
8.5.4 Positive solutions of elliptic and parabolic equations . . 304
8.6 Exercises 306

Chapter 9 Heteroclinic cycles 309
9.1 Examples and definitions 312
9.2 Asymptotic stability of a heteroclinic cycle 323
9.2.1 A Poincaré map for heteroclinic cycles 327
9.2.2 The proof of Theorem 9.2.2 330
9.2.3 Stability of heteroclinic cycles in $\mathbb{R}^4$ 334
9.3 Bifurcations from a heteroclinic cycle 340
9.3.1 The resonant bifurcation for a homoclinic cycle in $\mathbb{R}^3$. 342
9.3.2 A resonant bifurcation for homoclinic cycles in $\mathbb{R}^4$ . . . 344
9.3.3 The transverse bifurcation in $\mathbb{R}^4$ 347
9.4 Exercises 348

Chapter 10 Perturbation of equivariant systems 353
10.1 Asymmetric World 353
10.2 Perturbation of Group Orbits 357
10.3 Heteroclinic cycles by forced symmetry breaking 362
10.3.1 An example for breaking $\mathbf{D}_4 \ltimes \mathbb{T}^2$ to $\mathbf{D}_2$ 362
10.3.2 Breaking $\mathbf{O}(3)$-symmetry to $\mathbf{T}$-symmetry 367
10.3.3 Forced symmetry breaking of homoclinic cycles 369
10.4 Exercises 373

Appendix A Miscellanea on the group SO(3) 375
A.1 The subgroups of $\mathbf{O}(3)$ 375
A.2 Subconjugacy 378
A.3 Fixed point subspaces 380

Appendix B Translation table for the subgroups of O(3) 383

Bibliography 385

Index 401