

Contents

CHAPTER 0. Preliminaries		1
1. Introduction		1
2. Complex numbers		1
3. Functions		6
4. Polynomials		17
5. Complex series and the exponential function		20
6. Determinants		27
7. Remarks on methods of discovery and proof		30
CHAPTER 1. Introduction—Linear Equations of the First Order		33
1. Introduction		33
2. Differential equations		33
3. Problems associated with differential equations		36
4. Linear equations of the first order		39
5. The equation $y' + ay = 0$		39
6. The equation $y' + ay = b(x)$		40
7. The general linear equation of the first order		43
CHAPTER 2. Linear Equations with Constant Coefficients		49
1. Introduction		49
2. The second order homogeneous equation		50
3. Initial value problems for second order equations		54
4. Linear dependence and independence		60
5. A formula for the Wronskian		65
6. The non-homogeneous equation of order two		66
7. The homogeneous equation of order n		71
8. Initial value problems for n -th order equations		75
9. Equations with real constants		80
10. The non-homogeneous equation of order n		84
11. A special method for solving the non-homogeneous equation		90
12. Algebra of constant coefficient operators		94

CHAPTER 3. Linear Equations with Variable Coefficients	103
1. Introduction	103
2. Initial value problems for the homogeneous equation	104
3. Solutions of the homogeneous equation	106
4. The Wronskian and linear independence	111
5. Reduction of the order of a homogeneous equation	118
6. The non-homogeneous equation	122
7. Homogeneous equations with analytic coefficients	126
8. The Legendre equation	132
*9. Justification of the power series method	138
CHAPTER 4. Linear Equations with Regular Singular Points	143
1. Introduction	143
2. The Euler equation	145
3. Second order equations with regular singular points—an example	151
4. Second order equations with regular singular points—the general case	155
*5. A convergence proof	160
6. The exceptional cases	162
7. The Bessel equation	168
8. The Bessel equation (continued)	172
9. Regular singular points at infinity	180
CHAPTER 5. Existence and Uniqueness of Solutions to First Order Equations	185
1. Introduction	185
2. Equations with variables separated	186
3. Exact equations	192
4. The method of successive approximations	200
5. The Lipschitz condition	208
6. Convergence of the successive approximations	210
7. Non-local existence of solutions	217
8. Approximations to, and uniqueness of, solutions	222
9. Equations with complex-valued functions	227
CHAPTER 6. Existence and Uniqueness of Solutions to Systems and n-th Order Equations	229
1. Introduction	229
2. An example—central forces and planetary motion	231
3. Some special equations	236

4. Complex n -dimensional space 239
 5. Systems as vector equations 246
 6. Existence and uniqueness of solutions to systems 250
 7. Existence and uniqueness for linear systems 255
 8. Equations of order n 258

References 263

Answers to Exercises 265

Index 289

Preliminaries

1. Introduction

In this preliminary chapter we consider briefly some important concepts from calculus and algebra which we shall require for our study of differential equations. Many of these concepts may be familiar to the student, in which case this chapter can serve as a review. First the elementary properties of complex numbers are outlined. This is followed by a discussion of functions which assume complex values, in particular polynomials and power series. Some consequences of the Fundamental Theorem of Algebra are given. The exponential function is defined using power series; it is of great importance for linear differential equations with constant coefficients. The role that determinants play in the solution of systems of linear equations is discussed. Lastly we make a few remarks concerning principles of discovery, and methods of proof, of mathematical results.

1. Complex numbers

It is a fundamental fact about real numbers that the square of any such number is never negative. Thus there is no real x which satisfies the equation

$$x^2 + 1 = 0.$$

We shall use the real numbers to define new numbers which include numbers which satisfy such equations.

A complex number z is an ordered pair of real numbers (x, y) , and we write

$$z = (x, y).$$

$$z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2).$$