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A map between topological spaces is called *continuous* if the preimage of any open set is again open. A bijective map which is continuous in both directions is called a *homeomorphism*.

Definition 1.1.1 A manifold M of dimension d is a connected paracompact Hausdorff space for which every point has a neighborhood U that is homeomorphic to an open subset Ω of \mathbb{R}^d . Such a homeomorphism

$$x: U \rightarrow \Omega$$

is called a *(coordinate) chart*.

An *atlas* is a family $\{U_\alpha, x_\alpha\}$ of charts for which the U_α constitute an open covering of M .

Remarks.