

Contents

Preface	v
1. Introduction	1
1.1 Definition of Lie Algebras	1
1.2 Some Examples	2
1.3 Subalgebras and Ideals	3
1.4 Homomorphisms	4
1.5 Algebras	5
1.6 Derivations	6
1.7 Structure Constants	7
2. Ideals and Homomorphisms	11
2.1 Constructions with Ideals	11
2.2 Quotient Algebras	12
2.3 Correspondence between Ideals	14
3. Low-Dimensional Lie Algebras	19
3.1 Dimensions 1 and 2	20
3.2 Dimension 3	20
4. Solvable Lie Algebras and a Rough Classification	27
4.1 Solvable Lie Algebras	27
4.2 Nilpotent Lie Algebras	31
4.3 A Look Ahead	32

5. Subalgebras of $\mathfrak{gl}(V)$	37
5.1 Nilpotent Maps	37
5.2 Weights	38
5.3 The Invariance Lemma	39
5.4 An Application of the Invariance Lemma	42
6. Engel's Theorem and Lie's Theorem	45
6.1 Engel's Theorem	46
6.2 Proof of Engel's Theorem	48
6.3 Another Point of View	48
6.4 Lie's Theorem	49
7. Some Representation Theory	53
7.1 Definitions	53
7.2 Examples of Representations	54
7.3 Modules for Lie Algebras	55
7.4 Submodules and Factor Modules	57
7.5 Irreducible and Indecomposable Modules	58
7.6 Homomorphisms	60
7.7 Schur's Lemma	61
8. Representations of $\mathfrak{sl}(2, \mathbf{C})$	67
8.1 The Modules V_d	67
8.2 Classifying the Irreducible $\mathfrak{sl}(2, \mathbf{C})$ -Modules	71
8.3 Weyl's Theorem	74
9. Cartan's Criteria	77
9.1 Jordan Decomposition	77
9.2 Testing for Solvability	78
9.3 The Killing Form	80
9.4 Testing for Semisimplicity	81
9.5 Derivations of Semisimple Lie Algebras	84
9.6 Abstract Jordan Decomposition	85
10. The Root Space Decomposition	91
10.1 Preliminary Results	92
10.2 Cartan Subalgebras	95
10.3 Definition of the Root Space Decomposition	97
10.4 Subalgebras Isomorphic to $\mathfrak{sl}(2, \mathbf{C})$	97
10.5 Root Strings and Eigenvalues	99
10.6 Cartan Subalgebras as Inner-Product Spaces	102

11. Root Systems	109
11.1 Definition of Root Systems	109
11.2 First Steps in the Classification	111
11.3 Bases for Root Systems	115
11.4 Cartan Matrices and Dynkin Diagrams	120
12. The Classical Lie Algebras	125
12.1 General Strategy	126
12.2 $\mathfrak{sl}(\ell + 1, \mathbf{C})$	129
12.3 $\mathfrak{so}(2\ell + 1, \mathbf{C})$	130
12.4 $\mathfrak{so}(2\ell, \mathbf{C})$	133
12.5 $\mathfrak{sp}(2\ell, \mathbf{C})$	134
12.6 Killing Forms of the Classical Lie Algebras	136
12.7 Root Systems and Isomorphisms	137
13. The Classification of Root Systems	141
13.1 Classification of Dynkin Diagrams	142
13.2 Constructions	148
14. Simple Lie Algebras	153
14.1 Serre's Theorem	154
14.2 On the Proof of Serre's Theorem	158
14.3 Conclusion	160
15. Further Directions	163
15.1 The Irreducible Representations of a Semisimple Lie Algebra ...	164
15.2 Universal Enveloping Algebras	171
15.3 Groups of Lie Type	177
15.4 Kac–Moody Lie Algebras	179
15.5 The Restricted Burnside Problem	180
15.6 Lie Algebras over Fields of Prime Characteristic	183
15.7 Quivers	184
16. Appendix A: Linear Algebra	189
16.1 Quotient Spaces	189
16.2 Linear Maps	191
16.3 Matrices and Diagonalisation	192
16.4 Interlude: The Diagonal Fallacy	197
16.5 Jordan Canonical Form	198
16.6 Jordan Decomposition	200
16.7 Bilinear Algebra	201

17. Appendix B: Weyl's Theorem	209
17.1 Trace Forms	209
17.2 The Casimir Operator	211
18. Appendix C: Cartan Subalgebras	215
18.1 Root Systems of Classical Lie Algebras	215
18.2 Orthogonal and Symplectic Lie Algebras	217
18.3 Exceptional Lie Algebras	220
18.4 Maximal Toral Subalgebras	221
19. Appendix D: Weyl Groups	223
19.1 Proof of Existence	223
19.2 Proof of Uniqueness	224
19.3 Weyl Groups	226
20. Appendix E: Answers to Selected Exercises	231
Bibliography	247
Index	249