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of sets A_i , the intersection $A_1 \cap \dots \cap A_n$ or $\bigcap_{i=1}^n A_i$ of sets A_i , the difference $A - B$, and the cartesian product $A_1 \times \dots \times A_n$ of sets A_i . In particular, $A^n := A \times \dots \times A = \{(a_1, \dots, a_n) \mid a_i \in A\}$ is the set of n -tuples of elements of A . The cartesian product A^0 is defined to be $\{\emptyset\}$. The power set $\mathcal{P}(A)$ of a set A is the set of all subsets of A . We also consider the disjoint union $A_1 \sqcup \dots \sqcup A_n$ or $\biguplus_{i \in I} A_i$ of sets A_i , which may be realized as $\bigcup_{i \in I} (A_i \times \{i\})$. The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} respectively denote the sets of natural numbers (including 0), integers, positive integers, rationals, reals and complex numbers.

0.2 Relations. A subset of a cartesian product $A_1 \times \dots \times A_n$ is called a *relation* between the sets A_i . An n -ary relation on a set A is a subset of $A^n := A \times \dots \times A$. A 2-ary relation is called *binary*. We sometimes write $a \rho b$ to denote $(a, b) \in \rho$ for a binary relation ρ . The converse ρ^{-1} of a binary relation ρ on A is given by $(a, b) \in \rho^{-1}$ iff (i.e. if and only if) $(b, a) \in \rho$. A