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Banach spaces

Linear spaces

The basic ideas of this section go back to the Polish mathematician S. Banach who, by considering concrete spaces (sequence spaces, function spaces etc.), the definitions of this subsection are gained by abstraction.

Definition. A non-empty set M of elements is called a linear space if the following axioms are satisfied.

A. There is an operation called addition defined on M which assigns to any two elements x, y of M a unique element of M , denoted by $x + y$, in such a way that the following conditions (axioms) hold for all $x, y, z \in M$:

A.1. $x + y = y + x$ (commutativity).

A.2. $x + (y + z) = (x + y) + z$ (associativity).

A.3. There exists a unique neutral element θ (zero element) such that

$$x + \theta = x$$

for all elements x of M .

A.4. For every element x of M there exists exactly one element y of M (called the inverse element), such that

$$x + y = \theta$$

B. There is a second operation, called multiplication, which to every pair (λ, x) (where λ is a complex number and $x \in M$) assigns a unique element of M , denoted by $\lambda \cdot x$, in such a way that the following conditions (axioms) hold: