CONTENTS

PREFACE vii

CHAPTER I. PRELIMINARIES 1

1. Notation 1
2. Nature and purpose of differential geometry 2
3. Concept of mapping. Coordinates in Euclidean space 3
4. Vectors in Euclidean space 9
5. Basic rules of vector calculus in Euclidean space 11

CHAPTER II. THEORY OF CURVES 17

6. The concept of a curve in differential geometry 17
7. Further remarks on the concept of a curve 20
8. Examples of special curves 23
9. Arc length 25
10. Tangent and normal plane 29
11. Osculating plane 31
12. Principal normal, curvature, osculating circle 34
13. Binormal. Moving trihedron of a curve 36
14. Torsion 37
15. Formulae of Frenet 40
16. Motion of the trihedron, vector of Darboux 43
17. Spherical images of a curve 46
18. Shape of a curve in the neighbourhood of any of its points (canonical representation) 47
19. Contact, osculating sphere 49
20. Natural equations of a curve 55
21. Examples of curves and their natural equations 60
22. Involutes and evolutes 64
23. Bertrand curves 67

CHAPTER III. CONCEPT OF A SURFACE. FIRST FUNDAMENTAL FORM. FOUNDATIONS OF TENSOR-CALCULUS 72

24. Concept of a surface in differential geometry 72
25. Further remarks on the representation of surfaces, examples 76
26. Curves on a surface, tangent plane to a surface 79
27. First fundamental form. Concept of Riemannian geometry. Summation convention 82
28. Properties of the first fundamental form 85
29. Contravariant and covariant vectors 88
30. Contravariant, covariant, and mixed tensors 93

31. Basic rules of tensor calculus 99
32. Vectors in a surface. The contravariant metric tensor 101
33. Special tensors 105
34. Normal to a surface 107
35. Measurement of lengths and angles in a surface 109
36. Area 111
37. Remarks on the definition of area 115

CHAPTER IV. SECOND FUNDAMENTAL FORM. GAUSSIAN AND MEAN CURVATURE OF A SURFACE 118
38. Second fundamental form 118
39. Arbitrary and normal sections of a surface. Meusnier's theorem. Asymptotic lines 121
40. Elliptic, parabolic, and hyperbolic points of a surface 124
41. Principal curvature. Lines of curvature. Gaussian and mean curvature 128
42. Euler's theorem. Dupin's indicatrix 132
43. Torus 135
44. Flat points. Saddle points of higher type 136
45. Formulae of Weingarten and Gauss 138
46. Integrability conditions of the formulae of Weingarten and Gauss. Curvature tensors. Theorema egregium 142
47. Properties of the Christoffel symbols 148
48. Umbilics 152

CHAPTER V. GEODESIC CURVATURE AND GEODESICS 154
49. Geodesic curvature 154
50. Geodesics 157
51. Arcs of minimum length 160
52. Geodesic parallel coordinates 162
53. Geodesic polar coordinates 165
54. Theorem of Gauss–Bonnet. Integral curvature 168
55. Application of the Gauss–Bonnet theorem to closed surfaces 172

CHAPTER VI. MAPPINGS 175
56. Preliminaries 175
57. Isometric mapping. Bending. Concept of intrinsic geometry of a surface 176
58. Ruled surfaces, developable surfaces 179
59. Spherical image of a surface. Third fundamental form. Isometric mapping of developable surfaces 186
60. Conjugate directions. Conjugate families of curves. Developable surfaces contacting a surface 190
61. Conformal mapping 193
62. Conformal mapping of surfaces into a plane 195

63. Isotropic curves and isothermic coordinates 198
64. The Bergman metric 200
65. Conformal mapping of a sphere into a plane. Stereographic and Mercator projection 204
66. Equiareal mappings 208
67. Equiareal mapping of spheres into planes. Mappings of Lambert, Sanson, and Bonne 210
68. Conformal mapping of the Euclidean space 212

CHAPTER VII. ABSOLUTE DIFFERENTIATION AND PARALLEL DISPLACEMENT 219
69. Concept of absolute differentiation 219
70. Absolute differentiation of tensors of first order 220
71. Absolute differentiation of tensors of arbitrary order 223
72. Further properties of absolute differentiation 225
73. Interchange of the order of absolute differentiation. The Ricci identity 226
74. Bianchi identities 229
75. Differential parameters of Beltrami 229
76. Definition of the displacement of Levi–Cività 231
77. Further properties of the displacement of Levi–Cività 236
78. A more general definition of absolute differentiation and displacement of Levi-Cività 239

CHAPTER VIII. SPECIAL SURFACES 243
79. Definition and simple properties of minimal surfaces 243
80. Surfaces of smallest area 244
81. Examples of minimal surfaces 246
82. Relations between function theory and minimal surfaces. The formulae of Weierstrass 250
83. Minimal surfaces as translation surfaces with isotropic generators 253
84. Modular surfaces of analytic functions 255
85. Envelope of a one-parameter family of surfaces 262
86. Developable surfaces as envelopes of families of planes 268
87. Envelope of the osculating, normal, and rectifying planes of a curve, polar surface 270
88. Centre surfaces of a surface 273
89. Parallel surfaces 277
90. Surfaces of constant Gaussian curvature 279
91. Isometric mapping of surfaces of constant Gaussian curvature 280
92. Spherical surfaces of revolution 282
93. Pseudospherical surfaces of revolution 285
94. Geodesic mapping 290
95. Geodesic mapping of surfaces of constant Gaussian curvature 291
96. Surfaces of constant Gaussian curvature and non-Euclidean geometry 293

ANSWERS TO PROBLEMS 301

COLLECTION OF FORMULAE 328

BIBLIOGRAPHY 343

INDEX 348