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Behold, there stand the ensigns noble printed
 If you choose that wherein I am contain'd,
 Straight shall our nuptial rites be solemniz'd:
 But if you fail, without more speech, my lord,
 You must be gone from hence immediately.

William Shakespeare, Portia in *The Merchant of Venice*

1.1 Getting your goat

It's a critical moment for you. The master of ceremonies confronts you with three closed doors, one of which hides the car of your dreams, new and shiny and desirable. Behind each of the other two doors, however, is standing a pleasant but not so shiny and somewhat stinky goat. You will choose a door and win whatever is behind it. You decide on a door and announce your choice, whereupon the host opens one of the other two doors and reveals a goat. He then asks you if you would like to switch your choice to the unopened door that you did not at first choose. Is it to your advantage to switch (assuming, of course, that you are after the car, not the goat)?

This popular puzzler created a stir in 1991 when it appeared in the newspaper and (see [32], ¹) received a lot of wrong answers from readers, even from some who were mathematicians. How do we think about a problem

¹Numbers in square brackets refer to the references at the end of the book.