

Contents

| | | |
|---------------|--|-----------|
| 1 | Introduction: Principles and Tools of Mathematical Modeling | 1 |
| 1.1 | Role and Stages of Mathematical Modeling | 1 |
| 1.1.1 | Stages of Mathematical Modeling | 2 |
| 1.1.2 | Mathematical Modeling and Computer Simulation | 5 |
| 1.2 | Choice of Models | 6 |
| 1.2.1 | Deterministic and Stochastic Models | 6 |
| 1.2.2 | Continuous and Discrete Models | 7 |
| 1.2.3 | Linear and Nonlinear Models | 8 |
| 1.3 | Review of Selected Mathematical Tools | 9 |
| 1.3.1 | Derivatives and Integrals | 10 |
| 1.3.2 | Vector Algebra and Calculus | 12 |
| 1.3.3 | Differential Equations | 13 |
| 1.3.4 | Integral Equations | 16 |
| 1.3.5 | Optimization and Optimal Control | 19 |
| | Exercises | 21 |
| | References | 22 |
| | | |
| Part I | Mathematical Models in Economics | |
| 2 | Aggregate Models of Economic Dynamics | 25 |
| 2.1 | Production Functions and Their Types | 25 |
| 2.1.1 | Properties of Production Functions | 26 |
| 2.1.2 | Characteristics of Production Functions | 26 |
| 2.1.3 | Major Types of Production Functions | 27 |
| 2.1.4 | Two-Factor Production Functions | 29 |
| 2.2 | Solow–Swan Model of Economic Dynamics | 33 |
| 2.2.1 | Model Description | 34 |
| 2.2.2 | Analysis of Model | 35 |

| | | |
|----------|---|-----------|
| 2.3 | Optimization Versions of Solow–Swan Model | 37 |
| 2.3.1 | Optimization over Finite Horizon (Solow–Shell Model) | 38 |
| 2.3.2 | Infinite-Horizon Optimization (Solow–Ramsey Model) | 42 |
| 2.3.3 | Central Planner, General Equilibrium, and Nonlinear Utility | 44 |
| 2.4 | Appendix: Maximum Principle | 45 |
| 2.4.1 | Scalar Controls | 47 |
| 2.4.2 | Discounted Optimization | 47 |
| 2.4.3 | Interior Controls | 48 |
| 2.4.4 | Transversality Conditions | 49 |
| 2.4.5 | Maximum Principle and Dynamic Programming | 49 |
| | Exercises | 50 |
| | References | 51 |
| 3 | Modeling of Technological Change | 53 |
| 3.1 | Major Concepts of Technological Change | 53 |
| 3.1.1 | Exogenous Autonomous Technological Change | 54 |
| 3.1.2 | Embodied and Disembodied Technological Change | 55 |
| 3.1.3 | Endogenous Technological Change | 55 |
| 3.1.4 | Technological Change as Separate Sector of Economy | 56 |
| 3.2 | Models with Autonomous Technological Change | 56 |
| 3.2.1 | Solow–Swan Model | 58 |
| 3.2.2 | Solow–Shell Model | 60 |
| 3.2.3 | Solow–Ramsey Model | 62 |
| 3.3 | Models with Endogenous Technological Change | 63 |
| 3.3.1 | Induced Technological Change | 63 |
| 3.3.2 | One-Sector Model with Physical and Human Capital | 64 |
| 3.3.3 | Two-Sector Model with Physical and Human Capital (Uzawa–Lucas Model) | 68 |
| 3.3.4 | Knowledge-Based Models of Economic Growth | 70 |
| 3.4 | Modeling of Technological Innovations | 73 |
| 3.4.1 | Inventions, Innovations, and Spillovers | 73 |
| 3.4.2 | Substitution Models of Technological Innovations | 74 |
| 3.4.3 | Diffusion and Evolution Models of Technological Innovation | 76 |
| 3.4.4 | General Purpose Technologies and Technological Breakthroughs | 77 |
| | Exercises | 77 |
| | References | 78 |
| 4 | Models with Heterogeneous Capital | 79 |
| 4.1 | Macroeconomic Vintage Capital Models | 80 |
| 4.1.1 | Solow Vintage Capital Model | 80 |
| 4.1.2 | Vintage Models with Scrapping of Obsolete Capital | 82 |

| | | |
|--|--|------------|
| 4.1.3 | Two-Sector Vintage Model | 84 |
| 4.1.4 | Optimization Problems in Vintage Models | 85 |
| 4.2 | Vintage Capital Models of a Firm | 87 |
| 4.2.1 | Malcomson Model | 87 |
| 4.2.2 | Aggregate Production Functions | 90 |
| 4.3 | Vintage Models with Distributed Investments | 91 |
| 4.3.1 | Optimization Problems | 93 |
| 4.3.2 | Relations to Differential Models of Equipment Replacement | 94 |
| 4.4 | Discrete and Continuous Models of Machine Replacement | 96 |
| 4.4.1 | Multi-machine Replacement Model in Discrete Time | 96 |
| 4.4.2 | One-Machine Replacement in Discrete and Continuous Time | 99 |
| | Exercises | 101 |
| | References | 102 |
| 5 | Optimization of Economic Renovation | 105 |
| 5.1 | Optimal Replacement of One Machine | 105 |
| 5.1.1 | Necessary Condition for an Extremum | 106 |
| 5.1.2 | Qualitative Analysis of Optimal Replacement Policy | 107 |
| 5.2 | Profit-Maximizing Firm Under Resource Restrictions | 111 |
| 5.2.1 | Necessary Condition for an Extremum | 112 |
| 5.2.2 | Structure of Optimal Trajectories | 114 |
| 5.2.3 | Economic Interpretation | 116 |
| 5.3 | Nonlinear Utility Optimization in Ramsey Vintage Model | 117 |
| 5.3.1 | Reduction to One-Sector Optimization Problem | 118 |
| 5.3.2 | Interior Solutions | 120 |
| 5.3.3 | Balanced Growth | 120 |
| 5.3.4 | Economic Interpretation: Turnpike Properties | 122 |
| 5.4 | Appendix: Optimal Control in Vintage Capital Models | 123 |
| 5.4.1 | Statement of Optimization Problem | 124 |
| 5.4.2 | Variational Techniques | 125 |
| 5.4.3 | Method of Lagrange Multipliers | 126 |
| 5.4.4 | Extremum Conditions | 129 |
| | Exercises | 129 |
| | References | 130 |
| Part II Models in Ecology and Environment | | |
| 6 | Mathematical Models of Biological Populations | 133 |
| 6.1 | Models of Single Species Dynamics | 133 |
| 6.1.1 | Malthusian Growth Model | 134 |
| 6.1.2 | Von Bertalanffy Model | 135 |
| 6.1.3 | Verhulst–Pearl Model | 136 |
| 6.1.4 | Controlled Version of Verhulst–Pearl Model | 138 |
| 6.1.5 | Verhulst–Volterra Model with Hereditary Effects | 138 |

| | | |
|----------|--|------------|
| 6.2 | Models of Two Species Dynamics | 140 |
| 6.2.1 | Lotka–Volterra Model of Two Interacting Species | 140 |
| 6.2.2 | Lotka–Volterra Predator–Prey Model | 142 |
| 6.2.3 | Control in Predator–Prey Model | 145 |
| 6.2.4 | Generalized Predator–Prey Models | 146 |
| 6.2.5 | Predator–Prey Model with Individual Migration | 147 |
| 6.3 | Age-Structured Models of Population Dynamics | 149 |
| 6.3.1 | McKendrick Linear Population Model | 149 |
| 6.3.2 | MacCamy Nonlinear Population Model | 150 |
| 6.3.3 | Euler–Lotka Linear Integral Model of Population Dynamics | 151 |
| | Exercises | 153 |
| | References | 156 |
| 7 | Modeling of Heterogeneous and Controlled Populations | 157 |
| 7.1 | Linear Size-Structured Population Models | 157 |
| 7.1.1 | Model of Managed Size-Structured Population | 158 |
| 7.1.2 | Connection Between Age- and Size-Structured Models | 158 |
| 7.1.3 | Model of Size-Structured Population with Natural Reproduction | 159 |
| 7.2 | Nonlinear Population Models | 159 |
| 7.2.1 | Age-Structured Model with Intraspecies Competition | 160 |
| 7.2.2 | Bifurcation Analysis | 160 |
| 7.2.3 | Nonlinear Size-Structured Model | 162 |
| 7.2.4 | Steady-State Analysis | 163 |
| 7.3 | Population Models with Control and Optimization | 165 |
| 7.3.1 | Age-Structured Population Models with Control | 165 |
| 7.3.2 | Elements of Analysis | 167 |
| 7.3.3 | Nonlinear Age-Structured Models of Controlled Harvesting | 172 |
| 7.3.4 | Size-Structured Models with Controls | 173 |
| | Exercises | 175 |
| | References | 176 |
| 8 | Models of Air Pollution Propagation | 179 |
| 8.1 | Fundamentals of Environmental Pollutions | 179 |
| 8.2 | Models of Air Pollution Transport and Diffusion | 180 |
| 8.2.1 | Model of Pollution Transport | 181 |
| 8.2.2 | Model of Pollution Transport and Diffusion | 182 |
| 8.2.3 | Steady-State Analysis: One-Dimensional Stationary Distribution of Pollutant | 183 |
| 8.2.4 | Models of Pollution Transport, Diffusion, and Chemical Reaction | 184 |
| 8.2.5 | Control Problems of Pollution Propagation in Atmosphere | 185 |

| | | |
|----------|---|------------|
| 8.3 | Modeling of Plant Location | 185 |
| 8.3.1 | Adjoint Method | 186 |
| 8.4 | Control of Plant Pollution Intensity | 189 |
| 8.4.1 | Stationary Control of Air Pollution Intensity | 189 |
| 8.4.2 | Dynamic Control of Air Pollution Intensity | 191 |
| 8.5 | Structure of Applied Air Pollution Models | 192 |
| 8.5.1 | Interaction with Earth Surface | 193 |
| 8.5.2 | Interaction of Different Air Pollutants | 194 |
| 8.5.3 | Air Contamination in Cities | 194 |
| | Exercises | 195 |
| | References | 196 |
| 9 | Models of Water Pollution Propagation | 197 |
| 9.1 | Structure and Classification of Water Pollution Models | 197 |
| 9.1.1 | Structure of Models | 198 |
| 9.1.2 | Classification of Models | 198 |
| 9.2 | Three-Dimensional Model | 200 |
| 9.2.1 | Models of Adsorption and Sedimentation | 200 |
| 9.2.2 | Equation of Transport of Dissolved Pollutants | 201 |
| 9.2.3 | Equation of Transport of Suspended Pollutants | 202 |
| 9.2.4 | Equations of Surface Water Dynamics | 203 |
| 9.2.5 | Modeling of Pollutant Transport in Underground Water | 204 |
| 9.3 | Two-Dimensional Horizontal Model | 204 |
| 9.3.1 | Equation of Ground Deposit Accumulation | 204 |
| 9.3.2 | Equation of Transport of Dissolved Pollutants | 205 |
| 9.3.3 | Equation of Transport of Suspended Pollutants | 206 |
| 9.3.4 | Equations of Water Dynamics | 206 |
| 9.4 | One-Dimensional Pollution Model and Its Analytic Solutions | 207 |
| 9.4.1 | Link Between Convective Diffusion Equation and Heat Equation | 207 |
| 9.4.2 | Mathematical Preliminary: Heat Equation | 208 |
| 9.4.3 | Instantaneous Source of Pollutant | 209 |
| 9.4.4 | Pollutant Source with Constant Intensity | 210 |
| 9.5 | Compartmental Models and Control Problems | 212 |
| 9.5.1 | Equations of Water Balance | 212 |
| 9.5.2 | Equations of Suspension Balance | 212 |
| 9.5.3 | Equations of Pollution Propagation | 213 |
| 9.5.4 | Control Problems of Water Pollution Propagation | 214 |
| | Exercises | 215 |
| | References | 216 |

| | |
|---|-----|
| Part III Models of Economic-Environmental Systems | |
| 10 Modeling of Nonrenewable Resources | 221 |
| 10.1 Aggregate Models of Nonrenewable Resources | 221 |
| 10.1.1 Models of Optimal Resource Extraction | 222 |
| 10.1.2 Linear Model with No Resource Extraction Cost | 222 |
| 10.1.3 Models with Resource Extraction Cost | 224 |
| 10.1.4 Hotelling's Rule of Resource Extraction | 229 |
| 10.1.5 Modifications of Hotelling's Model | 231 |
| 10.1.6 Stochastic Models of Resource Extraction | 232 |
| 10.2 Dasgupta–Heal Model of Economic Growth with Exhaustible Resource | 234 |
| 10.2.1 Optimality Conditions | 235 |
| 10.2.2 Analysis of Model | 236 |
| 10.2.3 Interpretation of Results | 239 |
| Exercises | 239 |
| References | 240 |
| 11 Modeling of Environmental Protection | 241 |
| 11.1 Mutual Influence of Economy and Environment | 241 |
| 11.1.1 Climate Change and Environmental Strategies | 241 |
| 11.1.2 Modeling of Economic Impact on Environment | 243 |
| 11.1.3 Modeling of the Environmental Impact on Economy and Society | 244 |
| 11.1.4 Modeling of Mitigation and Adaptation | 246 |
| 11.2 Model with Pollution Emission and Abatement | 247 |
| 11.2.1 Optimality Conditions | 249 |
| 11.2.2 Analysis of Model | 249 |
| 11.2.3 Interpretation of Results | 251 |
| 11.3 Model with Pollution Accumulation and Abatement | 252 |
| 11.3.1 Analysis of Model | 252 |
| 11.3.2 Interpretation of Results | 254 |
| 11.4 Model with Pollution Abatement and Environmental Adaptation | 254 |
| 11.4.1 Optimality Conditions | 256 |
| 11.4.2 Steady-State Analysis | 257 |
| 11.4.3 Discussion of Results | 258 |
| Exercises | 260 |
| References | 261 |
| 12 Models of Global Dynamics: From Club of Rome to Integrated Assessment | 263 |
| 12.1 Global Trends and Their Modeling | 263 |
| 12.1.1 Global Environmental Trends | 264 |
| 12.1.2 Global Demographic Trends | 265 |
| 12.1.3 Population and Environment | 265 |

| | | |
|----------------|---|------------|
| Chapter 12.1.4 | Modeling of Global Change | 266 |
| Chapter 12.1.5 | Simplified Models of Human–Environmental Interaction | 267 |
| Chapter 12.1.6 | Aggregate Indicators in Global Models | 268 |
| 12.2 | Models of World Dynamics | 269 |
| 12.2.1 | Forrester Model | 270 |
| 12.2.2 | Meadows Models | 272 |
| 12.2.3 | Mesarovic–Pestel Model | 275 |
| 12.2.4 | Limitations of World Dynamics Models | 275 |
| 12.3 | Integrated Assessment Models: Structure and Results | 276 |
| 12.3.1 | Deterministic Models of Climate and Economy (DICE, RICE, WITCH) | 277 |
| 12.3.2 | Deterministic Energy–Economy Models (Global 2100, CETA, MERGE, ECLIPSE) | 278 |
| 12.3.3 | Scenario-Based Integrated Models (IMAGE, TARGETS) | 279 |
| 12.3.4 | Probabilistic Integrated Models (PAGE, ICAM) | 280 |
| 12.3.5 | Limitations of Integrated Assessment Models | 281 |
| 12.4 | Global Modeling: A Look Ahead | 281 |
| | Exercises | 282 |
| | References | 283 |
| Index | | 285 |

1.1 Role and Stages of Mathematical Modeling

Mathematical modeling is a vital component of scientific research and policy making. Its effectiveness has been proven for centuries. The modeling provides an explanation and prediction of the behavior of complex economic and environmental systems and helps to obtain new theoretical knowledge about the nature and society. The concept of the economic–environmental system assumes the influence of both the economy and environment on each other and the possibility of human control in the system [7]. The importance of modeling of such systems increases proportionally to the scale of human impact on the environment.

Mathematical modeling and computer simulation have a special place among scientific methods. The advantages of modeling as compared to experimentation are as follows:

- Universal availability and applicability of modeling tools.
- Low costs and short timeline of the modeling process.
- Multiple simulations on a wide range of model parameters (“what-if” analysis).
- Possibility of making various model modifications and improvements.
- Evading negative outcomes of real experiments, and others.